Title	FINITE DETERMINACY OF EQUIVARIANT MAP GERMS(C$^ \infty$ - maps and their Singularities)
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Citation	数理解析研究所講究録 (1985), 550: 172-183
Issue Date	1985-02
URL	http://hdl.handle.net/2433/98880
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
550巻 1985年 172-183

FINITE DETERMINACY OF EQUIVARIANT MAP GERMS

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Equivariant Contact Equivalence

Let $\mathbb{k}$ denote either $\mathbb{R}$ or $\mathbb{C}$ and "smooth" mean C^∞ or analytic according to context. Let V and W be finite dimensional $\mathbb{k}$-representations of a group G which is a compact Lie group if $\mathbb{k} = \mathbb{R}$ and a reductive complex Lie group if $\mathbb{k} = \mathbb{C}$. The space of (resp. equivariant) smooth map germs from $(V,0)$ to $(W,0)$ is denoted by $\mathcal{E}(V,W)$ (resp. $\mathcal{E}_G(V,W)$).

Definitions

(1) Let $\mathcal{K}_G$ denote the group of germs, H, of equivariant diffeomorphisms of $(V \times W,0)$ for which there exist germs of equivariant diffeomorphisms, h, of $(V,0)$ such that the following diagram commutes:-

$$\begin{array}{ccccc} (V,0) & \longrightarrow & (V\times W,0) & \longrightarrow & (V,0) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (V,0) & \longrightarrow & (V\times W,0) & \longrightarrow & (V,0) \end{array} .$$

(2) Two germs f,g in $\mathcal{E}_G(V,W)$ are $\mathcal{K}_G$ - equivalent if there exists (H,h) in $\mathcal{K}_G$ such that $H(h(x), f(h(x))) = (x, g(x))$, where x is a co-ordinate system for V.

(3) A germ, f, in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if there exists a positive integer, s, such that any g in $\mathcal{E}_G(V,W)$ with $j^s g(0) = j^s f(0)$ is $\mathcal{K}_G$ - equivalent to f.

If the actions of G on V and W are trivial then we have the usual definitions of $\mathcal{K}$ - equivalence and finite determinacy due to Mather. The above definition of $\mathcal{K}_G$ seems to be the most natural generalisation to equivariant map germs; there are analogous definitions of $\mathcal{R}_G$, $\mathcal{L}_G$, $\mathcal{A}_G$ and $\mathcal{C}_G$, but, for simplicity, only $\mathcal{K}_G$ finite determinacy will be considered in the present paper.

Criteria for Finite Determinacy

For f in $\mathcal{E}(V,W)$ denote by tf the map $\mathcal{E}(V,V) \to \mathcal{E}(V,W)$; $\alpha \mapsto Df.\alpha$

if f belongs to $\mathcal{E}_G(V,W)$ then tf restricts to a map $\mathcal{E}_G(V,V) \to \mathcal{E}_G(V,W)$. Mather defined the (extended) $\mathcal{K}$ - tangent space of f to be the subspace of $\mathcal{E}_G(V,W)$:-

$$T\mathcal{K}(f) := tf\,\mathcal{E}(V,V) + f^*m(W).\mathcal{E}(V,W)$$

where $m(W)$ is the maximal ideal of $\mathcal{E}(W)$, the ring of germs of smooth functions on $(W,0)$. If f belongs to $\mathcal{E}_G(V,W)$ the $\mathcal{K}_G$ - tangent space is defined analogously by:-

$$T\mathcal{K}_G(f) := tf\,\mathcal{E}_G(V,V) + (f^*m(W).\mathcal{E}(V,W))^G$$

where the notation $(\ .\ )^G$ is used to denote the fixed point set of the natural action of G. Clearly we have

$$T\mathcal{K}_G(f) = (T\mathcal{K}(f))^G.$$

Theorem 1 [2], [7]

A germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if $\dim._{k}\, \mathcal{E}_G(V,W) \,/\, T\mathcal{K}_G(f) < \infty$. □

An immediate corollary of this is that if f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}$-finitely determined then it is also $\mathcal{K}_G$ - finitely determined. In fact Wall [11] proved that if $j^sf(0)$ is $\mathcal{K}$ - sufficient, then it is also $\mathcal{K}_G$ - sufficient.

A second characterisation of finitely determined germs states that a germ is finitely determined if and only if it has a representative which is "stable" in a punctured neighbourhood of 0 (see Wall [12] for the non-equivariant case).

<u>Definition</u>

A germ f in $\mathcal{E}_G(V,W)$ is (infinitesimally) $\mathcal{K}_G$ - stable at 0 if $T\mathcal{K}_G(f) = \mathcal{E}_G(V,W)$. Stability of germs of equivariant maps $f : V \to W$ at points $x \neq 0$ can be defined similarly after restricting f to a slice transversal to the orbit $G.x$ at x, provided the orbit is closed in V.

By a closed orbit of the action of G on V we mean an orbit which is closed as a subset of V with its usual topology. If $|G|$ is finite or $k = \mathbb{R}$ and G is compact then every orbit of G on V is closed. However this is not true in the general complex case (consider, for example, the action of $G = \mathbb{C}^*$ on $\mathbb{C}^2$ given by $t.(x,y) = (t^{-1}x, ty)$ where t is an element of $\mathbb{C}^*$ and (x,y) of $\mathbb{C}^2$). In fact if $k = \mathbb{C}$ and G is a reductive complex Lie group with $\dim.G \geqslant 1$ then there always exist non-closed orbits. The necessity of the restriction to closed orbits in the definition is due to the fact that non-closed orbits do not, in general, possess suitable slices. Fortunately we only need a definition of stability at points on closed orbits.

<u>Theorem 2</u> [7]

Let $k = \mathbb{C}$. Then a germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if there exists a G - invariant neighbourhood U of 0 in V and a G - equivariant representative of f on U which is $\mathcal{K}_G$ - stable at every point in $U \setminus \{0\}$ for which the orbit $G.x$ is closed. □

If the actions of G on V and W are trivial then the theorem reduces to the usual statement that f is $\mathcal{K}$ - finitely determined if and only if a representative of f is $\mathcal{K}$- stable on $U \setminus \{0\}$ or, equivalently, if and only if it is transversal to 0 on $U \setminus \{0\}$.

Since a real analytic germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if its complexification $f_{\mathbb{C}}$ is $\mathcal{K}_{G_{\mathbb{C}}}$ - finitely determined (where $G_{\mathbb{C}}$ is the reductive complex Lie group obtained by complexifying the compact group G), this theorem also characterises $\mathcal{K}_G$ - finitely determined analytic germs when $k = \mathbb{R}$. Results analogous to Theorems 1 and 2 also hold for $\mathcal{R}_G, \mathcal{L}_G, \mathcal{A}_G$ and $\mathcal{C}_G$ finite determinacy.

Invariant Maps

Suppose the action of G on W is trivial; then the following lemma is not difficult to prove.

Lemma 3. An invariant map $f : V \longrightarrow W$ is $\mathcal{K}_G$- stable at a point x such that G.x is closed if and only if it is $\mathcal{K}$ - stable, and hence transversal to 0, at x. □

Corollary 4. If G is finite, then $f \in \mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if it is $\mathcal{K}$ - finitely determined.

The proof of the corollary is by the lemma, Theorem 2 and the fact that all the orbits of G in V are closed. □

If $\dim.G \geqslant 1$, and if $\dim.V > \dim.W > 1$, then Wall has shown [11] that no invariant map germ $f : V \to W$ with $Df(0) = 0$ can be $\mathcal{K}$ - finitely determined (Slodowy [10] proved a similar result for the case $\dim.W = 1$, but the situation when $\dim.V \leqslant \dim.W$ is unclear). However, although most invariant map germs are not $\mathcal{K}$ -finitely determined if $\dim.G \geqslant 1$, they are $\mathcal{K}_G$ -finitely determined. More precisely we have the following result .

<u>Theorem 5</u> [8]

For any G, if G acts trivially on W, then the germs in $\mathcal{E}_G(V,W)$ which are not $\mathcal{K}_G$ - finitely determined lie in a subset of infinite codimension. □

In a rather different direction we have the following proposition about real analytic germs.

<u>Proposition 6</u>

If $k = \mathbb{R}$ and f is a $\mathcal{K}_G$ - finitely determined real analytic invariant map germ then it is also C^k - $\mathcal{K}$ - finitely determined for all k such that $0 \leqslant k < \infty$.

<u>Proof</u>. If f is $\mathcal{K}_G$ - finitely determined there is a suitable representative $\tilde{f}$ of f such that $\tilde{f}_{\mathbb{C}}$ is transversal to 0 at all points with closed $G_{\mathbb{C}}$ - orbits in $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. If x is any point in V then $G_{\mathbb{C}}.x$ is closed (Schwarz [9] p.59) and so f is transversal to 0 at all points in a punctured neighbourhood of 0 in V. The result now follows from the geometric criterion for C^k - finite determinacy (see Wall [12] Theorem 6.1). □

Z_p - Equivariant Functions

If the action of G on W is non-trivial then we may lose the genericity, and even the existence, of $\mathcal{K}_G$ - finitely determined germs.

Let $G = Z_p$, identified with the p-th roots of unity; let t denote a generator of Z_p. Define an action of G on $V = \mathbb{C}^{m+n}$ by

$$t.(x_1,\ldots,x_m,y_1,\ldots,y_n) = (tx_1,\ldots,tx_m,y_1,\ldots,y_n)$$

and on $W = \mathbb{C}$ by

$$t.z = t^q z$$

for some q satisfying $0 \leqslant q < p$.

<u>Theorem 7</u> [8]

(1) There exist $\mathcal{K}_G$ - finitely determined germs in $\mathcal{E}_G(V,W)$ if and only if $n = 0$ or $n \geqslant \frac{(q+m-1)!}{q!(m-1)!} - m^2$

(2) Finite determinacy with respect to $\mathcal{K}_G$ holds outside a subset of infinite codimension if and only if one (or more) of the following conditions holds:-

(a) $n = 0$ (b) $q \leqslant 2$

(c) $m \leqslant 1$ (c) $(m,q) = (2,3),(2,4)$ or $(3,3)$.

We give a sketch of a proof of the theorem.

Any f in $\mathcal{E}_G(V,W)$ can be written

$$f(x,y) = \sum_j \bar{f}_j(x,y)\,\tau_j(x)$$

where $\{\tau_j\}$ is the set of monomials of degree q in $x_1,\ldots,x_m$ and the $\bar{f}_j$ are invariant functions on V. So f is determined by a (non-unique) invariant map $\bar{f} : V \longrightarrow M$, where M is the space of homogeneous forms of degree q in m variables.

Using arguments similar to some which are standard in singularity theory it can be shown that

f is $\mathcal{K}_G$ - stable at x in $V \setminus V^G$ if and only if it is transversal to 0 at x,

and

f is $\mathcal{K}_G$ - stable at x in V^G if and only if $\bar{f}|_{V^G} : V^G \longrightarrow M$ is transversal to the orbits of the natural action of $GL(m)$ on M.

Thus, by Theorem 2, $\mathcal{K}_G$ - finite determinacy is reduced to a set of transversality conditions on a punctured neighbourhood of 0.

The number of these transversality conditions is finite if and only if $n = 0$ or the number of $GL(m)$ orbits in M is finite; this latter condition is satisfied if and only if $m \leqslant 1$ or $q \leqslant 2$ or $(m,q) = (2,3)$. In these cases $\mathcal{K}_G$ finite determinacy is certainly generic.

More generally we can stratify M so that each stratum is foliated by $GL(m)$ orbits. If the strata of codimension $< n$ are foliated by orbits of codimension $\leqslant 1$ in the stratum then the points where $\bar{f}$ is not transversal to the $GL(m)$ orbits generically form a set of dimension 0 and so they can be excluded from a sufficiently small punctured neighbourhood of 0. Conversely, if there is a stratum of codimension $< n$ which is foliated by orbits of codimension > 1, then we can construct $\bar{f}$ (and hence f) with points of non-transversality in any neighbourhood of 0 which can not be removed by perturbing $\bar{f}$ slightly. This means that no extension of a suitable jet of f can be $\mathcal{K}_G$ - finitely determined, and so $\mathcal{K}_G$ - finite determinacy can not be generic. There exists a stratification of M such that the strata of codimension $< n$ are foliated by orbits of codimension $\leqslant 1$ if and only if we are in one of the cases of the previous paragraph or $(m,q) = (3,3)$ or $(2,4)$. This completes the proof of (2).

The necessity of the condition $n = 0$ or $n \geqslant \dim.M - m^2 = \frac{(q+m-1)!}{q!(m-1)!} - m^2$ in (1) is clear from the above since $\dim.M - m^2$ is the codimension of the orbits of maximal dimension in M. The sufficiency of this condition follows by more or less explicit construction of $\mathcal{K}_G$ - finitely determined germs. $\square$

Theorems 5 and 7 follow from a more general result [8] which gives necessary and sufficient conditions for $\mathcal{R}_G$, $\mathcal{C}_G$ and $\mathcal{K}_G$ finite determinacy to be generic in $\mathcal{E}_G(V,W)$ for any reductive complex Lie group G and complex representations V and W. There appears to be a serious problem with proving the necessity of the conditions when $k = \mathbb{R}$.

Stable Unfoldings

In complete analogy with the non-equivariant case Damon [2] proved that an equivariant map germ has a $\mathcal{K}_G$ - versal unfolding if and only if it is $\mathcal{K}_G$ - finitely determined (corresponding results hold for $\mathcal{R}_G$, $\mathcal{L}_G$, $\mathcal{A}_G$ and $\mathcal{C}_G$). However there is one aspect of unfolding theory which doesn't generalise in such a straightforward manner. Recall that for ordinary map germs an unfolding is $\mathcal{A}$ stable if and only if it is $\mathcal{K}$ - versal and so the property of possessing an $\mathcal{A}$ stable unfolding is generic; this latter fact is important in the proof of the topological stability theorem [4] and also in that of the genericity of topological $\mathcal{A}$ - finite determinacy [6].

However for equivariant map germs this is no longer true, instead we need to replace $\mathcal{K}_G$ by a subgroup, denoted $\mathcal{K}_G^*$ and defined in [13]. Then it is true that an equivariant unfolding is $\mathcal{A}_G$ - stable if and only if it is $\mathcal{K}_G^*$ - versal and so an equivariant germ has an $\mathcal{A}_G$ - stable unfolding if and only if it is $\mathcal{K}_G^*$ - finitely determined. If the action of G on W is trivial then $\mathcal{K}_G^* = \mathcal{K}_G$, but in general $\mathcal{K}_G^*$ finite determinacy is a stronger property than $\mathcal{K}_G$ - finite determinacy and $\mathcal{K}_G^*$ finite determinacy may not be generic even when $\mathcal{K}_G$ - finite determinacy is.

<u>Example.</u> Let $G = Z_2$ with generator t and V be the $m+n$ dimensional representation defined by:-

$$t.(x_1,\ldots,x_m,y_1,\ldots,y_n) = (tx_1,\ldots,tx_m,y_1,\ldots,y_n)$$

and W the similarly defined $p+q$ dimensional representation.

<u>Theorem 8</u> [8]

(1) Finite determinacy with respect to $\mathcal{K}_G$ is generic in $\mathcal{E}_G(V,W)$ for all m,n,p and q.

(2) Finite determinacy with respect to $\mathcal{K}_G^*$ is generic in $\mathcal{E}_G(V,W)$ if and only if one of the following holds:-

(a) $m = 0$ (b) $p \leq 2$

(c) $n \leq \begin{cases} m-p+q+1 & m \geq p \\ q & m < p \end{cases}$.

The non-genericity of $\mathcal{K}_G^*$ finite determinacy casts doubts on the possibility of an equivariant version of the topological stability theorem and Nakai [5] has found examples of pairs (M,N) of Z_2 - manifolds for which equivariant topologically stable maps are not dense and, locally, topological $\mathcal{A}_G$ - finite determinacy is not generic. It is perhaps interesting to note that in these examples the pairs of slice representations (V,W) of pairs of points (x,y) in $M^G \times N^G$ lie in the range $n = q+1$, $m = p-1$ and so $\mathcal{K}_G^*$ finite determinacy is not generic in $\mathcal{E}_G(V,W)$. It may be possible to prove an equivariant version of the topological stability theorem if $\mathcal{K}_G^*$ finite determinacy is generic for all possible pairs of slice representations; we might also conjecture that if $\mathcal{K}_G^*$ - finite determinacy is generic in $\mathcal{E}_G(V,W)$, then so is topological $\mathcal{A}_G$ - finite determinacy.

A Conjecture

So far we have no equivalence relation for which finite determinacy is generic for all G,V and W. However the following looks promising.

Definition

Two germs f,g in $\mathcal{E}_G(V,W)$ are C^0 - V_G - equivalent if there exists a germ of an equivariant homeomorphism of (V,0) taking $f^{-1}(0)$ to $g^{-1}(0)$.

Conjecture

The set of germs in $\mathcal{E}_G(V,W)$ which are not C^0 - V_G - finitely determined has infinite codimension.

A proof of this conjecture could perhaps be constructed using Thom's "blowing-up" idea (e.g. [6]) together with the equivariant transversality theory and isotopy theorem of Bierstone [1] and Field [3].

Finally we note that some related work, on the infinite determinacy of equivariant germs, has been done by Wall [14].

Acknowledgments

The results of the author contained in this paper are taken from his Ph.D. thesis which was supervised by C.T.C. Wall and supported by a SERC research studentship. The author is also grateful to SERC for their current financial support and to M. Oka and the Tokyo Institute of Technology for their hospitality during his visit to Japan.

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